MTH 201: Multivariable Calculus and Differential Equations

Homework I

(Due 19/08)

1. Let f be a scalar field defined on a set S, and let $c \in \mathbb{R}$. The set

$$L_c(f) = \{x \in S \mid f(x) = c\}$$

is called the *level set of f at c*. For each of the following, assume that S is the whole of $\mathbb{R}^n$, and sketch $L_c(f)$.

   (a) $f(x, y) = e^{xy}$, $c = e^{-2}, 1, e^3$.

   (b) $f(x, y, z) = \sin(x^2 + y^2 + z^2)$, $c = -\frac{1}{2}, \frac{\sqrt{2}}{2}, 1$.

2. In each of the following, let S be the set of all points in $\mathbb{R}^n$ satisfying the given inequalities. Determine whether S is open, closed, both open and closed, or neither open nor closed. Also determine ∂S

   (a) $xy < 1$.

   (b) $x \geq 0$ and $y > 0$.

   (c) $(x^2 + y^2 - 1)(4 - x^2 - y^2) > 0$.

   (d) $x + y + z < 1$ and $x > 0, y > 0, z > 0$.

   (e) $|x| \leq 1$, $|y| < 1$, and $|z| < 1$.

   (f) $1 < x^2 + y^2 \leq 2$

   (g) $y \geq x^2$ and $|x| \leq 2$.

   (h) $x^2 + y^2 \geq 0$.

3. Prove that the following sets are open in $\mathbb{R}^n$.

   (a) $\emptyset$ and $\mathbb{R}^n$.

   (b) The union of any arbitrary collection of open subsets.

   (c) The intersection of any finite collection of open subsets.

   (d) The product of any finite collection of open subsets.

   (e) The complement of any finite subset.

   (f) The interior and exterior of any subset.

4. Prove that the following sets are closed in $\mathbb{R}^n$.

   (a) The intersection of any arbitrary collection of closed subsets.

   (b) The union of any finite collection of closed subsets.

   (c) Any finite set.

   (d) The boundary of any subset. (Hint: Show that $\mathbb{R}^n = S^\circ \sqcup \mathrm{ext}(S) \sqcup \partial S$.)

5. Give counterexamples to disprove the following statements.

   (a) The intersection of any arbitrary collection of open subsets of $\mathbb{R}^n$ is open.

   (b) The union of any arbitrary collection of closed subsets $\mathbb{R}^n$ is closed.

6. If $S \subset \mathbb{R}^n$, then prove the following:

   (a) $\text{ext}(S) = (\mathbb{R}^n \setminus S)^\circ$.

   (b) S is closed if and only if $S = S^\circ \cup \partial S$.

7. For each of the following scalar fields, determine the set of points (x, y) such that f is continuous.

   (a) $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$.

   (b) $f(x, y) = x^{y^2}$.

   (c) $f(x, y) = \cos^{-1}\left(\sqrt{\frac{x}{y}}\right)$.

   (d) $f(x, y) = \dfrac{\cos(x^2)}{y}$.

8. For each of the following scalar fields, show that $\lim\limits_{x \to 0}[\lim\limits_{y \to 0} f(x, y)] \neq \lim\limits_{y \to 0}[\lim\limits_{x \to 0} f(x, y)]$.

   (a) $f(x, y) = \dfrac{x - y}{x + y}$, if $x \neq -y$.

   (b) $f(x, y) = \begin{cases} x \sin\left(\frac{1}{y}\right), & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$

9. For each of the following scalar fields, determine if it is possible to define $f(0, 0)$ so as to make f continuous at the origin.

   (a) $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$, if $(x, y) \neq (0, 0)$.

   (b) $f(x, y) = \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$, if $(x, y) \neq (0, 0)$.